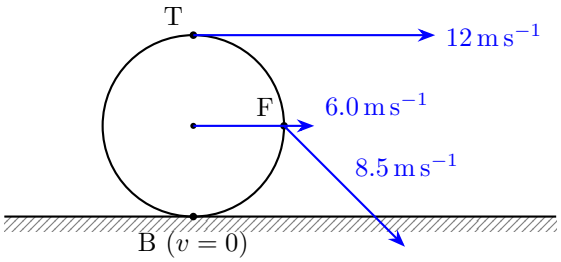
Tutorial Answers & Comments: Rotation of a Rigid Body

Part A: Rotational kinematics and moment of inertia

A1. (a) Same for P and Q: ω and α (every point on a rigid body turns through the same angle in the same time). Different: v, a_t, and a_r, which are all proportional to r and so are larger at Q (twice as large). (b) P: $\omega = 4.0\,\mathrm{rad\,s^{-1}}$, $v = 0.40\,\mathrm{m\,s^{-1}}$, $a_t = 0.20\,\mathrm{m\,s^{-2}}$, $a_r = 1.6\,\mathrm{m\,s^{-2}}$. Q: $\omega = 4.0\,\mathrm{rad\,s^{-1}}$, $v = 0.80\,\mathrm{m\,s^{-1}}$, $a_t = 0.40\,\mathrm{m\,s^{-2}}$, $a_r = 3.2\,\mathrm{m\,s^{-2}}$. (c) $\vec{v}$ is tangent to the circle, pointing in the CCW sense; $\vec{a}_t$ is parallel to $\vec{v}$ (the disk is speeding up); $\vec{a}_r$ points toward the axis. The total accelerations are $1.6\,\mathrm{m\,s^{-2}}$ (P) and $3.2\,\mathrm{m\,s^{-2}}$ (Q), tilted from the inward radial line toward the direction of motion by $\phi = \tan^{-1}(a_t/a_r) = \tan^{-1}(\alpha/\omega^2) = \tan^{-1}(0.125) = 7.1^\circ$ at both points, since the ratio α/ω^2 does not depend on r. Watch for students who point $\vec{a}$ along $\vec{v}$ or draw $\vec{a}_r$ outward. (d) Student 2. v is larger at Q only because r is larger: $v = \omega r$.

A2. (a) (i) $I = 2(0.50)(0.30)^2 = 0.090\,\mathrm{kg\,m^2}$. (ii) The mass on the axis contributes zero, and the other is at 0.60 m: $I = (0.50)(0.60)^2 = 0.18\,\mathrm{kg\,m^2}$. (iii) Both point masses lie on the axis, so $I = 0$. (b) $K = \frac{1}{2}I\omega^2$ with $\omega = 10\,\mathrm{rad\,s^{-1}}$: (i) 4.5 J; (ii) 9.0 J; (iii) 0 J. Axis (ii) is the hardest to start spinning, since it has the largest I: more mass is far from the axis. (Axis (iii) is an idealization; real spheres have a small nonzero I.) (c) Student 2. Same masses, same rod, but I changes by a factor of 2 between (i) and (ii), and goes to 0 for (iii): I measures how the mass is distributed relative to the *chosen axis*. (d) $I_{\mathrm{cm}} + Md^2 = 0.090 + (1.0)(0.30)^2 = 0.18\,\mathrm{kg\,m^2}$, which matches (ii). This is the parallel-axis theorem.

A3. (a) $\omega = v_{\mathrm{cm}}/R = 6.0/0.35 = 17\,\mathrm{rad\,s^{-1}}$, clockwise (the wheel rolls to the right). The no-slip relation is $v_{\mathrm{cm}} = \omega R$. (b) Relative to the center, every point on the rim moves at $\omega R = 6.0\,\mathrm{m\,s^{-1}}$: T to the right, B to the left, F downward. (c) Relative to the ground, add $\vec{v}_{\mathrm{cm}} = 6.0\,\mathrm{m\,s^{-1}}$ to the right: T: $12\,\mathrm{m\,s^{-1}}$ to the right; C: $6.0\,\mathrm{m\,s^{-1}}$ to the right; B: zero; F: $\sqrt{6.0^2 + 6.0^2} = 8.5\,\mathrm{m\,s^{-1}}$, directed 45° below the horizontal (forward and down). B is momentarily at rest.



(d) Student 2. Spinning alone would give B a velocity of $\omega R = 6.0\,\mathrm{m\,s^{-1}}$ backward, but the center's motion adds $6.0\,\mathrm{m\,s^{-1}}$ forward, so $v_B = 0$. With no slipping, the contact point does not slide, so the friction is static. Since the point of application does not move, static friction does zero work on a rolling wheel. Many students think "friction always does negative work"; ask what the displacement of the contact point is.

Part B: Torque and rotational dynamics

B1. (a) I: $r = 0.80\,\text{m}$, $\phi = 90°$, $\tau = +8.0\,\text{N m}$ (CCW). II: $r = 0.40\,\text{m}$, $\phi = 90°$, but $\vec{F}$ points down, so $\tau = -4.0\,\text{N m}$ (CW). III: $r = 0.80\,\text{m}$, $\phi = 30°$, $\tau = (0.80)(10)\sin 30° = +4.0\,\text{N m}$ (CCW). IV: $r = 0.80\,\text{m}$, $\phi = 0°$, $\tau = 0$. (b) $\tau_{\text{net}} = 8.0 - 4.0 + 4.0 + 0 = +8.0\,\text{N m}$, so the rod would start to rotate CCW (the force from the pivot has zero torque about the pivot, so it does not enter). (c) Student 2. Same force and same point of application, but III has only the component $F\sin 30° = 5.0\,\text{N}$ perpendicular to the rod, which is half of I. Case IV: a force along the line from the pivot has zero torque; it can only stretch the rod. (d) $\tau = r_\perp F$: need $F\,(0.40) = 8.0$, so $F = 20\,\text{N}$.

B2. (a) Block: weight $m\vec{g}$ (Earth, down) and tension $\vec{T}$ (string, up). Pulley: tension $\vec{T}$ (string, down, at the rim), weight $M\vec{g}$ (Earth, at the axle), and the support force from the axle (up, at the axle). Only the tension exerts a torque about the axle, $\tau = TR$; the weight and the axle force act at the axis and have zero lever arm. (b) Student 2. If $T = mg$ the block would have no acceleration, so the pulley could not accelerate either; but the unbalanced torque TR must spin it up. The tension must be smaller than mg so that the block can accelerate downward: $mg - T = ma$. (c) Block: $mg - T = ma$. Pulley: $TR = I\alpha = \frac{1}{2}MR^2\alpha$. No-slip: $a = \alpha R$. Then $T = \frac{1}{2}Ma$, and $mg = (m + \frac{1}{2}M)a$, so $a = \dfrac{(2.0)(9.8)}{2.0 + 2.0} = 4.9\,\text{m s}^{-2}$, $T = \frac{1}{2}(4.0)(4.9) = 9.8\,\text{N}$ (half of $mg = 19.6\,\text{N}$), $\alpha = a/R = 20\,\text{rad s}^{-2}$. (d) $v = \sqrt{2ah} = \sqrt{2(4.9)(1.5)} = 3.8\,\text{m s}^{-1}$, and $\omega = v/R = 15\,\text{rad s}^{-1}$. Energy: the block loses $mgh = 29\,\text{J}$. $K_{\text{block}} = \frac{1}{2}(2.0)(14.7) = 14.7\,\text{J}$; $K_{\text{pulley}} = \frac{1}{2}(0.125)(15.3)^2 = 14.7\,\text{J}$. These sum to 29.4 J, as they should; for a solid disk the pulley gets exactly half of the energy because $\frac{1}{2}I\omega^2 = \frac{1}{4}Mv^2$ and $M = 2m$ here.

B3. Prediction: with $I = MR^2$ (twice as large as the disk's), the pulley is harder to spin up, so a is smaller and the string has to pull harder, so T is larger (closer to mg). Then $TR = MR^2\alpha$ gives $T = Ma$, so $a = \dfrac{mg}{m + M} = \dfrac{19.6}{6.0} = 3.3\,\text{m s}^{-2}$ and $T = (4.0)(3.27) = 13\,\text{N}$ ($< mg = 19.6\,\text{N}$, but larger than 9.8 N). Limiting cases: for $M \to 0$, $T \to 0$ and $a \to g$ (a massless pulley cannot support a tension if it has a torque on it); for $M \to \infty$, $a \to 0$ and $T \to mg$.

★B4. (a) The tensions are *not* equal: the net torque on the pulley is $(T_1 - T_2)R = I\alpha \neq 0$, so $T_1 > T_2$ (the heavier side pulls harder). (b) $m_1 g - T_1 = m_1 a$, $T_2 - m_2 g = m_2 a$, $(T_1 - T_2)R = \frac{1}{2}MR^2(a/R)$, which gives $a = \dfrac{(m_1 - m_2)g}{m_1 + m_2 + M/2} = \dfrac{(2.0)(9.8)}{5.0} = 3.9\,\text{m s}^{-2}$. $T_1 = 3.0(9.8 - 3.92) = 17.6\,\text{N}$; $T_2 = 1.0(9.8 + 3.92) = 13.7\,\text{N}$. (c) $T_1 - T_2 = 3.92\,\text{N}$ and $\frac{1}{2}Ma = 1.0(3.92) = 3.92\,\text{N}$, which agree.

★B5. There is no external torque about the axle (gravity acts parallel to the axle, and the axle exerts no torque), so $L = I\omega$ is conserved. $I_{\text{disk}} = \frac{1}{2}(2.0)(0.30)^2 = 0.090\,\text{kg m}^2$, $I_{\text{clay}} = (0.50)(0.30)^2 = 0.045\,\text{kg m}^2$. $\omega_f = \dfrac{(0.090)(6.0)}{0.135} = 4.0\,\text{rad s}^{-1}$. $K_i = \frac{1}{2}(0.090)(36) = 1.62\,\text{J}$, $K_f = \frac{1}{2}(0.135)(16) = 1.08\,\text{J}$, so 0.54 J (one third) is lost. This is the rotational analog of a completely inelastic collision: the lost fraction is $I_{\text{clay}}/(I_{\text{disk}} + I_{\text{clay}}) = 1/3$, in the same way that the lost fraction in last week's collisions was $m_B/(m_A + m_B)$.

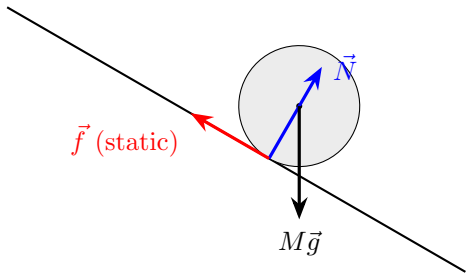
Part C: Guided practice: a rolling race down an incline

C1. Typical predictions: "the heaviest wins" or "the biggest wins". Do not correct them yet. The answer (from C5) is that the order is sphere, cylinder, hoop, and it does not depend on M or R.

C2. (a) $K = \frac{1}{2}Mv^2 + \frac{1}{2}(cMR^2)(v/R)^2 = \frac{1}{2}Mv^2(1+c)$. (b) $Mgh = \frac{1}{2}(1+c)Mv^2$, so $v = \sqrt{\dfrac{2gh}{1+c}}$; M and R cancel. Static friction acts at a point that is instantaneously at rest, so it does no work and does not heat anything. (Friction *is* necessary, though: see C4.) (c) Rotational fraction $c/(1+c)$: cylinder $\frac{1}{3}$, hoop $\frac{1}{2}$, sphere $\frac{2}{7}$.

C3. $2gh = 11.76\,\mathrm{m^2\,s^{-2}}$. Cylinder: $v = \sqrt{11.76/1.5} = 2.8\,\mathrm{m\,s^{-1}}$. Hoop: $v = \sqrt{11.76/2} = 2.4\,\mathrm{m\,s^{-1}}$. Sphere: $v = \sqrt{11.76/1.4} = 2.9\,\mathrm{m\,s^{-1}}$. Frictionless block: $v = \sqrt{2gh} = 3.4\,\mathrm{m\,s^{-1}}$. The rolling objects are slower because part of the energy Mgh goes into rotational kinetic energy, leaving less for translation. The more mass far from the axis (larger c), the larger the rotational share.

C4. (a) Friction points *up* the incline. Without it, the object would slide with the gravitational pull, and the contact point would move down the slope relative to the surface; friction opposes that sliding tendency. It also provides the torque that makes the object turn CW (to roll down the incline). Students who write "down the incline" are using "friction opposes the motion of the center." (b) On a frictionless incline, the object just slides without rotating (no torque about the center), and $v = 3.4\,\mathrm{m\,s^{-1}}$ for all three. (c) Only friction exerts a torque about the center (lever arm R). The weight acts at the center, and the line of action of the normal force passes through the center of a round object, so both have zero torque.



C5. (a) $Mg\sin\theta - f = Ma$. (b) $fR = I\alpha = cMR^2\alpha$, so $f = cM\alpha R = cMa$. (c) $Mg\sin\theta = Ma + cMa$, so $a = \dfrac{g\sin\theta}{1+c}$. (d) a: cylinder $\dfrac{(9.8)(0.5)}{1.5} = 3.3\,\mathrm{m\,s^{-2}}$; hoop $2.5\,\mathrm{m\,s^{-2}}$; sphere $3.5\,\mathrm{m\,s^{-2}}$; frictionless block $g\sin\theta = 4.9\,\mathrm{m\,s^{-2}}$. Times from $d = \frac{1}{2}at^2$ with $d = 1.2\,\mathrm{m}$: cylinder 0.86 s; hoop 0.99 s; sphere 0.83 s; block 0.70 s. Check: $v^2 = 2ad = 2(3.27)(1.2) = 7.84$, so $v = 2.8\,\mathrm{m\,s^{-1}}$, in agreement with the energy result in C3. So the order is sphere, cylinder, hoop; mass and radius cancel. Ask groups who predicted "heavier wins" to point to where M cancelled.

★C6. From (a) and (b): $f = cMa = \dfrac{c}{1+c}Mg\sin\theta$. Requiring $f \le \mu_s Mg\cos\theta$ gives $\mu_s \ge \dfrac{c\tan\theta}{1+c}$. With $\tan 30^\circ = 0.577$: cylinder 0.19; hoop 0.29; sphere 0.16. The hoop needs the roughest incline, since it needs the largest friction to supply its larger torque requirement (I is larger). If μ_s is smaller than this, the object slips as it goes and $v_{\mathrm{cm}} \neq \omega R$.

★C7. The solid cylinder ($c = \frac{1}{2}$) beats the thin hollow shell ($c \approx 1$) every time: $a = 3.3$ vs $2.5\,\mathrm{m\,s^{-2}}$. Release the two together from rest and see which one gets to the bottom first, or time each one. The answer does not depend on the mass or radius, only on how the mass is distributed (the dimensionless c), which is why this works even though the two cylinders have different densities.