Differential Error Propagation

The method I descibe here is the answer to the following question:

> ``If I measure quantities
>
> $a \pm \Delta a, b \pm \Delta b, c \pm \Delta c, \mathrm{etc}\ldots$
>
> how do I compute the error in the **computed** quantity
>
> $$y = f(a, b, c)$$
>
> for some *arbirary* function f?``

We discussed this in class with a simple motivating example. The general result we wrote down is

> $$\Delta y = \left\{ (\frac{\partial f}{\partial a}\Delta a)^2 + (\frac{\partial f}{\partial b}\Delta b)^2 + (\frac{\partial f}{\partial c}\Delta c)^2 + \ldots \right\}^{\frac{1}{2}} \quad (1)$$
>
> where $\partial f / \partial a$ is the partial derivative of the function f with respect to a. There is a special case where the above general expressions simplifies enormously: if
>
> $$y = k a^m b^n c^p$$
>
> then
>
> $$\frac{\Delta y}{y} = \left\{ \left(m \frac{\Delta a}{a} \right)^2 + \left(n \frac{\Delta b}{b} \right)^2 + \left(p \frac{\Delta c}{c} \right)^2 \right\}^{\frac{1}{2}} \quad (2)$$

It's important to note that the result (eq. 2) is *only* valid if y is a *product* of powers. For example, if

$$y = A + k a^m b^n c^p$$

where A is some other measured quantity, then---unfortunately for you---you will have to use equation 1, to compute the uncertainty in y.

Exercises for Lab 1.

Please write you answers **neatly** on a separate page. Remember to show your work, explaining your reasoning step-to-step with text *and* equations. If there is a numerical result requested, put a neat box around your final numerical answer(s). Questions 1 through 3 are each worth 3 points. You get 1 point for being human, and 1 extra credit point possible on top of the max total of 10 points. 11 points with the extra credit. Neatness counts in my evaluation.

1. Suppose that you measure an acceleration $a \pm \Delta a$, and a time interval $t \pm \Delta t$, and you compute a distance $y = \frac{1}{2}at^2$. What should you report for your experimental error $y \pm \Delta y$?

   a. Write an algebraic expression for $y \pm \Delta y$.

   b. Evaluate your expression if $a = 2.4 \pm 0.2\ \text{m/s}^2$ and $t = 10.0 \pm 0.1\ \text{s}$

   c. Which of the measured quantities contributes most to the uncertainty in y? Explain.

2. Suppose that you measure charges $q_1 \pm \Delta q_1$ and $q_2 \pm \Delta q_2$ as well as a separation distance $r \pm \Delta r$, and you compute the electical force each charge exerts on the other via Coulomb's Law:

   $F = \frac{k\, q_1\, q_2}{r^2}$

   a. Write an algebraic expression for $F \pm \Delta F$. Write

   b. Evaluate your expression if $q_1 = (2.0 \pm 0.2)\ \mu\text{C}$, $q_2 = (80.0 \pm 0.2)\ \mu\text{C}$ and $r = (3.0 \pm 0.2)\ \mu\text{m}$

   c. Rank the contributions to the uncertianty in order of increasing contribution.

3. Suppose that you measure an amplitude $A \pm \Delta A$ (where A is in meters) and a time $t \pm \Delta t$ (t in seconds). Suppose you are given the value of an angular velocity ω (in rad/s), assumed constant with zero uncertainty. Write an algebraic expression for the uncertainty in the quantity $x = A\cos(\omega t)$. You will have to use equation 1 and compute two partial derivatives. Don't forget about the need (in one of those derivatives) to use the chain rule.

4. **Extra Credit**: Prove that equation 2 follows from equation 1.